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Generalized minimal dominating graphical indices

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Abstract

The significance of the generalized minimal dominating graphical indices is that their specific cases for randomly chosen values of the non-zero real numbers m and n, which are coincide with the vast majority of pre-defined graphical indices being considered. In this paper, we obtain some specific families of graphs, bounds and characterization in terms of order, size, minimum / maximum dominating degree and other dominating degree-based graphical indices. Also, we present the chemical applicability of molecular graph of some basic Benzenoid structures of above said graphical indices.

Keywords: Domination degree, Minimal dominating set, Domination Zagreb indices, Total number of minimal domination set.

2020 MSC: 05C09, 05C69, 05C92.

1. Introduction

By a graph $G = (V(G), E(G))$, we mean a finite, undirected and simple graph. As usual $p = |V(G)|$ and $q = |E(G)|$ denotes the number of vertices and edges of (p, q)-graph G, respectively. Let $deg(v)$ be the degree of vertex v and as usual $\delta(G) = \delta$, the minimum degree, and $\Delta(G) = \Delta$, the maximum degree of G. A graph G is r-regular if $\delta = \Delta = r$. The induced subgraph $\langle X \rangle$ is the subgraph of G with the vertex set X. The open neighborhood $N(v)$ of vertex v denotes the set of vertices adjacent to v and its closed neighborhood $N[v] = N(v) \cup \{v\}$. For graph-theoretical terminology, we refer to [14]. A set $D \subseteq V(G)$ is a dominating set of G if every vertex in $V(G) - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. Further, a dominating set D is a minimal dominating set if no proper subset of D is a dominating set of G.

Observation 1. If D is a minimal dominating set, then for every vertex $v \in D$, there is a vertex $u \in N[v]$ which is dominated only by v. We will call such a vertex u, a private neighbor of v, since u is not adjacent to any vertex in $D - \{v\}$.

Observation 2. Every minimum dominating set is a minimal dominating set, but the converse is not true in general, one such example is the graph $G \cong S_{1,s}$, where $S_{1,s}$ is a star with $(s + 1)$-vertices.

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For a complete review of the theory of domination and its related parameters, we refer to [5], [16]-[19] and [23].

In the present research work, the value of each vertex $v \in V(G)$, called the dominating degree of v denoted by $\gamma_G(v)$, and is defined along with the number of minimal dominating sets of G which contains v. The minimum and maximum dominating degree of G are denoted by $\delta_d(G) = \delta_d$ and $\Delta_d(G) = \Delta_d$, respectively. Further, a graph G is r_d-minimal dominating regular if $\delta_d = \Delta_d = r_d$. Also, the total number of minimal dominating sets of a graph G are denoted by $T_{MD}(G)$. This concept was initiated by Ahmed et al., [1]-[4] and studied by Basavangoud et al., [6] and Kante et al., [25].

Observation 3. For any non-trivial simple graph G,

$$1 \leqslant \gamma_G(v) \leqslant T_{MD}(G).$$

The use of graphical indices has been extensively studied. For their history, applications, and mathematical properties, see [8]-[13] and the references cited therein.

In this paper, we initiate the novel generalization of minimal dominating graphical indices of a graph G with two real numbers m and n are as follows:

(i) The (m, n)-sum minimal dominating index of G is

$$SMD_{(m,n)}(G) = \sum_{uv \in E(G)} [\gamma_G^m(u) + \gamma_G^m(v)]^n.$$

(ii) The (m, n)-product minimal dominating index of G is

$$PMD_{(m,n)}(G) = \sum_{uv \in E(G)} [\gamma_G^m(u).\gamma_G^m(v)]^n.$$

(iii) The (m, n)-difference minimal dominating index of G is

$$DMD_{(m,n)}(G) = \sum_{uv \in E(G)} |\gamma_G^m(u) - \gamma_G^m(v)|^n.$$

2. Specific cases for randomly chosen values of m and n

The majority of hitherto studied dominating degree-based graphical indices are special cases of (m, n)-minimal dominating graphical indices of a graph G, for particular values of non-zero real numbers m and n are as shown in below Table 1.

(m, n)-minimal dominating graphical indices	Name of the graphical indices
$SMD_{(1,1)}(G) = DM_1^*(G)$	The modified first Zagreb dominating index, [1]
$SMD_{(2,1)}(G) = DF^*(G)$	The modified Forgotten dominating index, [3]
$SMD_{(1,2)}(G) = DH(G)$	The hyper dominating index, [3]
$SMD_{(1,-1)}(G) = Dh(G)$	The dominating Harmonic index, [1]
$SMD_{(1,\frac{1}{2})}(G) = DN(G)$	The dominating Nirmala index, [20]
$SMD_{(1,-\frac{1}{2})}(G) =^m DN(G)$	The modified dominating Nirmala index, [20]
$SMD_{(-1,-1)}(G) =^m DM_2^*(G)$	The modified second Zagreb dominating index, [2]
$SMD_{(2,\frac{1}{2})}(G) = DSO(G)$	The Sombor dominating index, [26]
$PMD_{(1,1)}(G) = DM_2(G)$	The second dominating Zagreb index, [1]
$PMD_{(1,\frac{1}{2})}(G) = RDP(G)$	The Reciprocal dominating product connectivity index, [22]
$PMD_{(1,-\frac{1}{2})}(G) = DP(G)$	The dominating product connectivity index, [22]
$DMD_{(1,\frac{1}{2})}(G) = IDN(G)$	An irregularity dominating Nirmala index, [21]
$DMD_{(2,\frac{1}{2})}(G) = IDSO(G)$	An irregularity dominating Sombor index, [21]
$DMD_{(1,2)}(G) = D_\sigma(G)$	The dominating Sigma index, [28]

Table 1: The particular values of (m, n)-minimal dominating graphical indices.

3. Some specific families of graphs

Here, computed values of some specific families of graphs are presented without proof.

Proposition 3.1. For any complete graph K_p with $p \geqslant 3$,

(i) $SMD_{(m,n)}(K_p) = 2^{n-1}\, p(p-1)$.
(ii) $PMD_{(m,n)}(K_p) = \dfrac{p(p-1)}{2}$.
(iii) $DMD_{(m,n)}(K_p) = 0$.

Proposition 3.2. For any complete bipatite graph $K_{t,s}$ with $2 \leqslant t \leqslant s$,

(i) $SMD_{(m,n)}(K_{t,s}) = ts\,\big((t+1)^m + (s+1)^m\big)^n$.
(ii) $PMD_{(m,n)}(K_{t,s}) = ts\,\big((t+1)(s+1)\big)^{mn}$.
(iii) $DMD_{(m,n)}(K_{t,s}) = \begin{cases} ts\big|(t+1)^m - (s+1)^m\big|^n; & t < s \\ 0\,; & t = s. \end{cases}$

Corollary 3.3. For any star $S_{1,s}$ with $(s+1)$-vertices for $s \geqslant 1$,

(i) $SMD_{(m,n)}(S_{1,s}) = 2^{mn}s$.
(ii) $PMD_{(m,n)}(S_{1,s}) = s^{mn}$.
(iii) $DMD_{(m,n)}(S_{1,s}) = 0$.

Proposition 3.4. For any double star graph $S_{t,s}$ with $t \geqslant 2$ and $s \geqslant 3$,

(i) $SMD_{(m,n)}(S_{t,s}) = (t+s-1)\, 2^{n(m+1)}$.
(ii) $PMD_{(m,n)}(S_{t,s}) = (t+s-1)\, 4^{mn}$.
(iii) $DMD_{(m,n)}(S_{t,s}) = 0$.

4. Bounds in terms of order, size, degree domination and total number of minimal dominating set

Theorem 4.1. Let G be a (p,q)-graph with $p \geqslant 2$. Then

(i) $2^{mn}q \leqslant SMD_{(m,n)}(G) \leqslant 2^{mn}\, q\, (p-1)^{mn}$.
(ii) $q \leqslant PMD_{(m,n)}(G) \leqslant q(p-1)^{2mn}$.
(iii) $0 \leqslant DMD_{(m,n)}(G) \leqslant q((p-1)^m - 1)^n$.

Proof. Let G be a (p,q)-graph with $p \geqslant 2$. If $1 \leqslant \{\gamma_G(u), \gamma_G(v)\} \leqslant p-1$, then

(i) $$2 \leqslant \gamma_G(u) + \gamma_G(v) \leqslant 2(p-1)$$

$$2^m \leqslant \gamma_G^m(u) + \gamma_G^m(v) \leqslant 2^m\, (p-1)^m$$
$$2^{mn} \leqslant [\gamma_G^m(u) + \gamma_G^m(v)]^n \leqslant 2^{mn}(p-1)^{mn}.$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$\sum_{uv\in E(G)} 2^{mn} \leqslant \sum_{uv\in E(G)} [\gamma_G^m(u) + \gamma_G^m(v)]^n \leqslant \sum_{uv\in E(G)} 2^{mn}(p-1)^{mn}.$$

Therefore, $2^{mn}\, q \leqslant SMD_{(m,n)}(G) \leqslant 2^{mn}\, q\, (p-1)^{mn}$.

(ii) $$1 \leqslant \gamma_G(u).\gamma_G(v) \leqslant (p-1)^2$$

$$1 \leqslant \gamma_G^m(u).\gamma_G^m(v) \leqslant (p-1)^{2m}$$
$$1 \leqslant [\gamma_G^m(u).\gamma_G^m(v)]^n \leqslant (p-1)^{2mn}.$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$\sum_{uv \in E(G)} 1 \leqslant \sum_{uv \in E(G)} [\gamma_G^m(u).\gamma_G^m(v)]^n \leqslant \sum_{uv \in E(G)} (p-1)^{2mn}.$$

Therefore, $q \leqslant PMD_{(m,n)}(G) \leqslant q(p-1)^{2mn}$.

(iii) Similarly, we have $0 \leqslant DMD_{(m,n)}(G) \leqslant q((p-1)^m - 1)^n$. □

Theorem 4.2. Let G be a (p, q)-graph with $p \geqslant 2$. Then

(i) $2^n q\delta_d^{mn}(G) \leqslant SMD_{(m,n)}(G) \leqslant 2^n q\Delta_d^{mn}(G)$
(ii) $q\delta_d^{2mn}(G) \leqslant PMD_{(m,n)}(G) \leqslant q\Delta_d^{2mn}(G)$
(iii) $0 \leqslant DMD_{(m,n)}(G) \leqslant q|(\Delta_d^m(G) - \delta_d^m(G))|^n$.

The lower and upper bounds holds if and only if G is r_d-minimal dominating regular.

Proof. Let G be a (p, q)-graph with $p \geqslant 2$. If $\delta_d(G) \leqslant \{\gamma_G(u), \gamma_G(v)\} \leqslant \Delta_d(G)$, then

(i) $$2\delta_d^m(G) \leqslant \gamma_G^m(u) + \gamma_G^m(v) \leqslant 2\Delta_d^m(G)$$

$$2^n\delta_d^{mn}(G) \leqslant [\gamma_G^m(u) + \gamma_G^m(v)]^n \leqslant 2^n\Delta_d^{mn}(G).$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$2^n q\delta_d^{mn}(G) \leqslant SMD_{(m,n)}(G) \leqslant 2^n q\Delta_d^{mn}(G).$$

(ii) $$\delta_d^{2m}(G) \leqslant \gamma_G^m(u).\gamma_G^m(v) \leqslant \Delta_d^{2m}(G)$$

$$\delta_d^{2mn}(G) \leqslant [\gamma_G^m(u).\gamma_G^m(v)]^n \leqslant \Delta_d^{2mn}(G).$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$q\delta_d^{2mn}(G) \leqslant PMD_{(m,n)}(G) \leqslant q\Delta_d^{2mn}(G).$$

(iii) Similarly, we have $0 \leqslant DMD_{(m,n)}(G) \leqslant q|(\Delta_d^m(G) - \delta_d)^m(G)|^n$.
The lower and upper bounds holds if and only if G is r_d-minimal dominating regular. □

Theorem 4.3. Let G be a (p, q)-graph with $p \geqslant 2$. Then

(i) $2^{mn}q \leqslant SMD_{(m,n)}(G) \leqslant 2^{mn}\ q\ T_{MD}^{mn}(G)$.
(ii) $q \leqslant PMD_{(m,n)}(G) \leqslant q\ T_{MD}^{2mn}(G)$.

Proof. Let G be a (p, q)-graph with $p \geqslant 2$. If $1 \leqslant \{\gamma_G(u), \gamma_G(v)\} \leqslant T_{MD}(G)$, then

(i) $$2 \leqslant \gamma_G(u) + \gamma_G(v) \leqslant 2T_{MD}(G)$$

$$2^m \leqslant \gamma_G^m(u) + \gamma_G^m(v) \leqslant 2^m T_{MD}^m(G)$$

$$2^{mn} \leqslant [\gamma_G^m(u) + \gamma_G^m m(v)]^n \leqslant 2^{mn}\ T_{MD}^{mn}(G).$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$2^{mn}q \leqslant SMD_{(m,n)}(G) \leqslant 2^{mn}\ q\ T_{MD}^{mn}(G).$$

(ii) $1 \leqslant \gamma_G^m(u).\gamma_G^m(v) \leqslant T_{MD}^{2m}(G)$

$$1 \leqslant [\gamma_G^m(u).\gamma_G^m(v)]^n \leqslant T_{MD}^{2mn}(G).$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$q \leqslant PMD_{(m,n)}(G) \leqslant q\ T_{MD}^{2mn}(G).$$

Hence, the proof is complete. □

Theorem 4.4. Let G be a (p, q)-graph with $p \geqslant 2$. Then

$$\frac{2^n}{\Delta_d^{mn}(G)} PMD_{(m,n)}(G) \leqslant SMD_{(m,n)}(G) \leqslant \frac{2^n}{\delta_d^{mn}(G)} PMD_{(m,n)}(G).$$

The lower and upper bounds holds if and only if G is r_d-minimal dominating regular.

Proof. Let G be a (p, q)-graph with $p \geqslant 2$ vertices. Then

$$[\gamma_G(u)^m + \gamma_G(v)^m]^n = \gamma_G^{mn}(u).\gamma_G^{mn}(v)\left[\frac{1}{\gamma_G^m(u)} + \frac{1}{\gamma_G^m(v)}\right]^m$$

$$\gamma_G(u)^{mn}\gamma_G(v)^{mn}\left[\frac{2}{\Delta_d^m(G)}\right]^n \leqslant [\gamma_G^m(u) + \gamma_G^m(v)]^n$$
$$\leqslant \gamma_G(u)^{mn}\gamma_G(v)^{mn}\left[\frac{2}{\delta_d^m(G)}\right]^n.$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$\sum_{uv\in E(G)} \gamma_G(u)^{mn}\gamma_G(v)^{mn}\left[\frac{2}{\Delta_d^m(G)}\right]^n \leqslant \sum_{uv\in E(G)} [\gamma_G^m(u) + \gamma_G^m(v)]^n$$
$$\leqslant \sum_{uv\in E(G)} \gamma_G(u)^{mn}\gamma_G(v)^{mn}\left[\frac{2}{\delta_d^m(G)}\right]^n.$$

$$\frac{2^n}{\Delta_d^{mn}(G)} PMD_{(m,n)}(G) \leqslant SMD_{(m,n)}(G) \leqslant \frac{2^n}{\delta_d^{mn}(G)} PMD_{(m,n)}(G).$$

The lower and upper bounds holds if and only if G is r_d-minimal dominating regular. □

Theorem 4.5. Let G be a (p, q)-graph with $p \geqslant 2$. Then

$$0 \leqslant DMD_{(m,n)}(G) \leqslant \left[\frac{1}{\delta_d^m(G)} - \frac{1}{\Delta_d^m(G)}\right]^n PMD_{(m,n)}(G).$$

The lower and upper bounds holds if and only if G is r_d-minimal dominating regular.

Proof. Let G be a (p, q)-graph with $p \geqslant 2$. Then

$$|\gamma_G^m(u) - \gamma_G^m(v)|^n = \gamma_G^{mn}(u).\gamma_G^{mn}(v)\left[\frac{1}{\gamma_G^m(u)} - \frac{1}{\gamma_G^m(v)}\right]^n.$$

$$0 \leqslant |\gamma_G^m(u) - \gamma_G^m(v)|^n \leqslant \gamma_G^{mn}(u).\gamma_G^{mn}(v)\left[\frac{1}{\delta_d^m(G)} - \frac{1}{\Delta_d^m(G)}\right]^n.$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$0 \leqslant \sum_{uv \in E(G)} |\gamma_G^m(u) - \gamma_G^m(v)|^n \leqslant \sum_{uv \in E(G)} \gamma_G^{mn}(u).\gamma_G^{mn}(v) \left[\frac{1}{\delta_d^m(G)} - \frac{1}{\Delta_d^m(G)}\right]^n.$$

$$0 \leqslant DMD_{(m,n)}(G) \leqslant \left[\frac{1}{\delta_d^m(G)} - \frac{1}{\Delta_d^m(G)}\right]^n PMD_{(m,n)}(G).$$

The lower and upper bounds holds if and only if G is r_d-minimal dominating regular. □

Theorem 4.6. Let G be a (p, q)-graph with $p \geqslant 2$. Then

(i) $DF^*(G) = DH(G) - 2DM_2(G)$.
(ii) $D\sigma(G) = DF^*(G) - 2DM_2(G)$.
(iii) $D\sigma(G) = DH(G) - 4DM_2(G)$.

Proof. Let G be a (p, q)-graph with $p \geqslant 2$.

(i) Consider
$$\begin{aligned} DF^*(G) &= \sum_{uv \in E(G)} [\gamma_G^2(u) + \gamma_G^2(v)] \\ &= \sum_{uv \in E(G)} [[\gamma_G(u) + \gamma_G(v)]^2 - 2\gamma_G(u)\gamma_G(v)] \\ &= \sum_{uv \in E(G)} [\gamma_G(u) + \gamma_G(v)]^2 - 2 \sum_{uv \in E(G)} [\gamma_G(u)\gamma_G(v)] \\ &= DH(G) - 2DM_2(G). \end{aligned}$$

(ii) Consider
$$\begin{aligned} D\sigma(G) &= \sum_{uv \in E(G)} |\gamma_G(u) - \gamma_G(v)|^2 \\ &= \sum_{uv \in E(G)} \left[\gamma_G(u)^2 + \gamma_G(v)^2 - 2\gamma_G(u)\gamma_G(v)\right] \\ &= DF^*(G) - 2DM_2(G). \end{aligned}$$

(iii) Consider
$$\begin{aligned} D\sigma(G) &= \sum_{uv \in E(G)} |\gamma_G(u) - \gamma_G(v)|^2 \\ &= \sum_{uv \in E(G)} \big(\gamma_G(u) + \gamma_G(v)\big)^2 - 4\gamma_G(u)\gamma_G(v) \\ &= DH(G) - 4DM_2(G). \end{aligned}$$

Hence, the proof is complete. □

To prove our next few results we make use of the following inequalities such as Harmonic mean, Geometric mean, Arithmatic mean and Quadratic mean (HM-GM-AM-QM) [7] as follows:

$$\frac{2xy}{x+y} \leqslant \sqrt{xy} \leqslant \frac{x+y}{2} \leqslant \sqrt{\frac{x^2+y^2}{2}}, \tag{4.1}$$

where x and y are non-zero real numbers.

Theorem 4.7. Let G be a (p, q)-graph with $p \geqslant 2$ vertices. Then

(i) $2RDP(G) \leqslant SMD_{(1,1)}(G) \leqslant \sqrt{2}DSO(G)$.
(ii) $4ISI(G) \leqslant SMD_{(1,1)}(G) \leqslant \sqrt{2}DSO(G)$.

(iii) $\frac{2}{3}[DM_1^*(G) + DN(G)] \leqslant SMD_{(1,1)}(G) \leqslant DF^*(G)Dh(G)$.

Proof. Let G be a (p, q)-graph with $p \geqslant 2$ vertices.

(i) By the definition of $SMD_{(m,n)}(G)$ and equation 4.1, we have

$$\sqrt{\gamma_G(u).\gamma_G(v)} \leqslant \frac{\gamma_G(u) + \gamma_G(v)}{2} \leqslant \frac{\sqrt{\gamma_G^2(u) + \gamma_G^2(v)}}{\sqrt{2}}.$$

$$2\sqrt{\gamma_G(u).\gamma_G(v)} \leqslant \gamma_G(u) + \gamma_G(v) \leqslant \sqrt{2}\sqrt{\gamma_G^2(u) + \gamma_G^2(v)}.$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$2\sum_{uv\in E(G)} \sqrt{\gamma_G(u).\gamma_G(v)} \leqslant \sum_{uv\in E(G)} \gamma_G(u) + \gamma_G(v) \leqslant \sqrt{2}\sum_{uv\in E(G)} \sqrt{\gamma_G^2(u) + \gamma_G^2(v)}.$$

Therefore, $2RDP(G) \leqslant SMD_{(1,1)}(G) \leqslant \sqrt{2}DSO(G)$.

(ii) By the definition of $SMD_{(m,n)}(G)$ and equation 4.1, we have

$$2\frac{\gamma_G(u).\gamma_G(v)}{\gamma_G(u) + \gamma_G(v)} \leqslant \frac{\gamma_G(u) + \gamma_G(v)}{2} \leqslant \sqrt{\frac{\gamma_G^2(u) + \gamma_G^2(v)}{2}}.$$

$$4\frac{\gamma_G(u).\gamma_G(v)}{\gamma_G(u) + \gamma_G(v)} \leqslant \gamma_G(u) + \gamma_G(v) \leqslant \sqrt{2}\sqrt{\gamma_G^2(u) + \gamma_G^2(v)}.$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$4\sum_{uv\in E(G)} \frac{\gamma_G(u).\gamma_G(v)}{\gamma_G(u) + \gamma_G(v)} \leqslant \sum_{uv\in E(G)} \gamma_G(u) + \gamma_G(v) \leqslant \sum_{uv\in E(G)} \sqrt{2}\sqrt{\gamma_G^2(u) + \gamma_G^2(v)}.$$

Therefore, $4ISI(G) \leqslant SMD_{(1,1)}(G) \leqslant \sqrt{2}DSO(G)$.

(iii) By the definition of $SMD_{(m,n)}(G)$ and equation 4.1, we have

$$\frac{\gamma_G(u) + \gamma_G(v)}{3} + \frac{\sqrt{\gamma_G(u).\gamma_G(v)}}{3} \leqslant \frac{\gamma_G(u) + \gamma_G(v)}{2} \leqslant \frac{\gamma_G^2(u) + \gamma_G^2(v)}{\gamma_G(u) + \gamma_G(v)}.$$

$$\frac{2}{3}[\gamma_G(u) + \gamma_G(v) + \sqrt{\gamma_G(u).\gamma_G(v)}] \leqslant \gamma_G(u) + \gamma_G(v) \leqslant (\gamma_G^2(u) + \gamma_G^2(v))\left(\frac{2}{\gamma_G(u) + \gamma_G(v)}\right).$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$\frac{2}{3}\sum_{uv\in E(G)} [\gamma_G(u) + \gamma_G(v) + \sqrt{\gamma_G(u).\gamma_G(v)}] \leqslant \sum_{uv\in E(G)} \gamma_G(u) + \gamma_G(v) \leqslant \sum_{uv\in E(G)} (\gamma_G^2(u) + \gamma_G^2(v))\left(\frac{2}{\gamma_G(u) + \gamma_G(v)}\right).$$

Therefore, $\frac{2}{3}[DM_1^*(G) + DN(G)] \leqslant SMD_{(1,1)}(G) \leqslant DF^*(G)Dh(G)$.

Hence, the proof is complete. □

To prove our next result, we make use of the definition of the dominating inverse sum indeg index of a graph G, and is defined as

$$DISI(G) = \sum_{uv \in E(G)} \frac{\gamma_G(u).\gamma_G(v)}{\gamma_G(u) + \gamma_G(v)}.$$

Theorem 4.8. Let G be a (p, q)-graph with $p \geqslant 2$. Then

$$2DISI(G) \leqslant PMD_{(1,\frac{1}{2})} \leqslant \frac{1}{2} DM_1^*(G).$$

Proof. We know that,

$$2\frac{\gamma_G(u).\gamma_G(v)}{\gamma_G(u) + \gamma_G(v)} \leqslant \left[\gamma_G(u).\gamma_G(v)\right]^{\frac{1}{2}} \leqslant \frac{\gamma_G(u) + \gamma_G(v)}{2}.$$

Apply summation for each edge $e = uv \in E(G)$, we have

$$2 \sum_{uv \in E(G)} \frac{\gamma_G(u).\gamma_G(v)}{\gamma_G(u) + \gamma_G(v)} \leqslant \sum_{uv \in E(G)} \left[\gamma_G(u).\gamma_G(v)\right]^{\frac{1}{2}}$$
$$\leqslant \frac{1}{2} \sum_{uv \in E(G)} \gamma_G(u) + \gamma_G(v).$$

Therefore, $2DISI(G) \leqslant PMD_{(1,\frac{1}{2})} \leqslant \frac{1}{2} DM_1^*(G)$. □

5. Chemical Applicabilities for Benzenoid structures

In 1865, German chemist August Kekule visualized the ring structure called Benzene (C_6H_6). Most chemical organic compounds contain a loop of six carbon atoms called Benzene rings. Benzene is a widely used industrial chemical and is a major part of gasoline. Some other uses of Benzene include making plastics, synthetic fibers, rubber lubricants, dyes, resins, detergents, drugs, and more. For more information, we refer to [15], [24] and [27].

In our study, we considered the molecular graph of some basic Benzenoid structures as shown in Figure 1. The dominating degree of each vertex v of Benzenoid structures such as Benzene, Naphthalene, Anthracene, and Phenanthrene is shown in Table 2. The computed values of the (m, n)-minimal dominating graphical

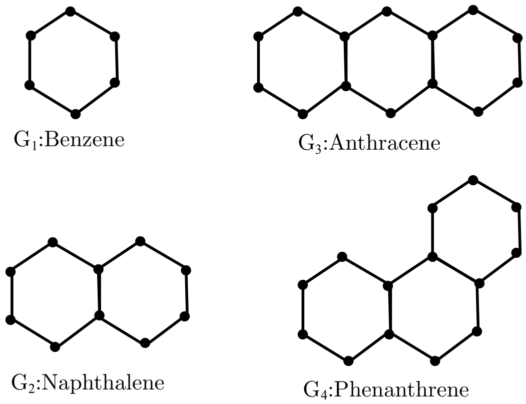

Figure 1: Molecular graph of some basic Benzenoid structures.

indices of the molecular graph of some basic Benzenoid structures as shown in Table 3. Further, the particular values of (m, n)-minimal dominating graphical indices of the molecular graph of some basic Benzenoid structures as shown in Table 4.

Molecular graphs	Vertex dominating degree													
	$\gamma_G(v_1)$	$\gamma_G(v_2)$	$\gamma_G(v_3)$	$\gamma_G(v_4)$	$\gamma_G(v_5)$	$\gamma_G(v_6)$	$\gamma_G(v_7)$	$\gamma_G(v_8)$	$\gamma_G(v_9)$	$\gamma_G(v_{10})$	$\gamma_G(v_{11})$	$\gamma_G(v_{12})$	$\gamma_G(v_{13})$	$\gamma_G(v_{14})$
G_1	2	2	2	2	2	2	-	-	-	-	-	-	-	-
G_2	8	6	6	8	6	6	7	8	6	5	-	-	-	-
G_3	12	12	6	11	12	6	14	8	9	14	6	14	7	8
G_4	11	9	9	5	11	10	5	14	10	7	8	12	5	12

Table 2: Vertex dominating degree of molecular graphs.

Molecular graphs	The computed values of (m,n)-minimal dominating indices		
	$SMD_{m,n}$	$PMD_{m,n}$	$DMD_{m,n}$
G_1	$6\ 2^n$	6	0
G_2	$2^{n+1}\ 6^{mn}+4[8^m+6^m]^n+2[6^m+5^m]^n$ $+[6^m+7^m]^n+[7^m+8^m]^n+[5^m+8^m]^n$	$4\ (48)^{mn}+2\ 6^{2mn}+2\ (30)^{mn}+(42)^{mn}$ $+(56)^{mn}+(40)^{mn}$	$4\lvert 8^m-6^m\rvert^n+2\lvert 6^m-5^m\rvert^n$ $+\lvert 6^m-7^m\rvert^n+\lvert 7^m-8^m\rvert^n+\lvert 5^m-8^m\rvert^n$
G_3	$2^n\ 12^{mn}+2^{n+1}14^{mn}+3\ [6^m+14^m]^n$ $+2[12^m+6^m]^n+2[8^m+12^m]^n+[6^m+11^m]^n$ $+[11^m+12^m]^n+[14^m+8^m]^n+[8^m+9^m]^n$ $+[9^m+14^m]^n+[14^m+7^m]^n+[7^m+8^m]^n$	$3\ 84^{mn}+2\ 96^{mn}+2\ 72^{mn}+12^{2mn}+14^{2mn}$ $+66^{mn}+132^{mn}+112^{mn}+72^{mn}+126^{mn}$ $+98^{mn}+56^{mn}$	$3\ \lvert 6^m-14^m\rvert^n+2\lvert 12^m-6^m\rvert^n+2\lvert 8^m-12^m\rvert^n$ $+\lvert 6^m-11^m\rvert^n+\lvert 11^m-12^m\rvert^n+\lvert 14^m-8^m\rvert^n+\lvert 8^m-9^m\rvert^n$ $+\lvert 9^m-14^m\rvert^n+\lvert 14^m-7^m\rvert^n+\lvert 7^m-8^m\rvert^n.$
G_4	$2\ [12^m+5^m]^n+[11^m+9^m]^n+2\ 2^n\ 9^{mn}$ $+[9^m+5^m]^n+[5^m+11^m]^n+[11^m+10^m]^n$ $+[10^m+5^m]^n+[5^m+14^m]^n+[14^m+10^m]^n$ $+[10^m+7^m]^n+[7^m+8^m]^n+[8^m+12^m]^n$ $+[12^m+11^m]^n+[12^m+9^m]^n+[8^m+10^m]^n$	$56^{mn}+12^{mn}[8^{mn}+2\ 5^{mn}+9^{mn}]+11^{mn}[9^{mn}$ $+12^{mn}+5^{mn}]+10^{mn}[11^{mn}+5^{mn}+2\ 7^{mn}$ $+14^{mn}+8^{mn}]+9^{mn}[5^{mn}+9^{mn}]$	$2\ \lvert 12^m-5^m\rvert^n+\lvert 11^m-9^m\rvert^n+\lvert 9^m-5^m\rvert^n$ $+\lvert 10^m-5^m\rvert^n+\lvert 5^m-14^m\rvert^n+\lvert 14^m-10^m\rvert^n$ $+\lvert 5^m-11^m\rvert^n+\lvert 11^m-10^m\rvert^n$ $+\lvert 10^m-7^m\rvert^n+\lvert 7^m-8^m\rvert^n+\lvert 8^m-12^m\rvert^n$ $+\lvert 12^m-11^m\rvert^n+\lvert 12^m-9^m\rvert^n+\lvert 8^m-10^m\rvert^n$

Table 3: The computed values of (m,n)-minimal dominating indices of molecular graphs.

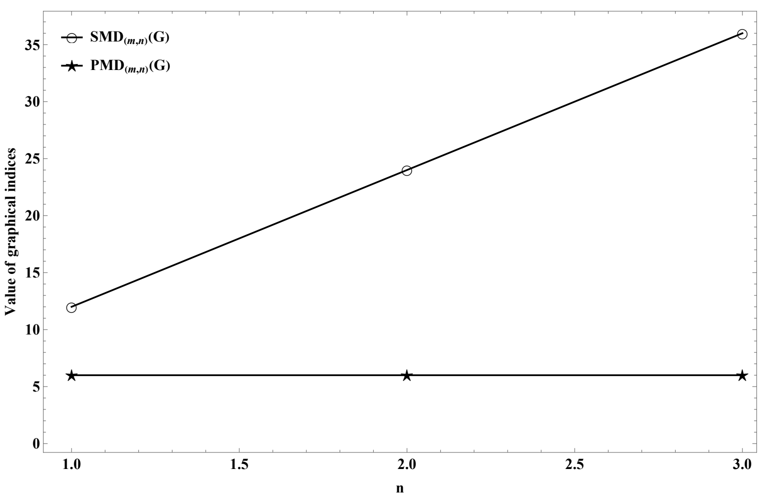

Figure 2: The comparative analysis of G_1.

Molecular graphs		$SMD_{(m,n)}$			$PMD_{(m,n)}$			$DMD_{(m,n)}$		
	m/n	1	2	3	1	2	3	1	2	3
G_1	1	12	24	36	6	6	6			
	2	12	24	36	6	6	6			
	3	12	24	36	6	6	6			
G_2	1	143	1877	2486×10^1	462	2010×10^1	9033×10^2	15	29	63
	2	953	8572×10^1	7962×10^3	2010×10^1	4171×10^4	2326×10^{11}	201	5293	1553×10^2
	3	6509	4175×10^3	2842×10^6	9033×10^2	9515×10^7	1173×10^{13}	2049	5614×10^2	1700×10^5
G_3	1	354	7578	1668×10^2	1590	1772×10^2	2229×10^4	70	434	2908
	2	4006	1036×10^3	2919×10^5	1772×10^2	3139×10^6	8091×10^{10}	1390	1734×10^2	2346×10^4
	3	4819×10^1	1622×10^4	6275×10^8	2229×10^4	8091×10^{10}	4794×10^{17}	2158×10^1	4298×10^4	9358×10^7
G_4	1	313	5889	1131×10^2	1312	1204×10^2	1220×10^4	59	317	2021
	2	3103	6098×10^2	1283×10^5	1204×10^2	1338×10^6	1885×10^{10}	1061	1018×10^2	1176×10^4
	3	3252×10^1	7239×10^4	1827×10^8	1220×10^4	1885×10^{10}	3918×10^{16}	1511×10^1	2142×10^4	3766×10^7

Table 4: The particular values of (m, n)-minimal dominating indices of molecular graphs.

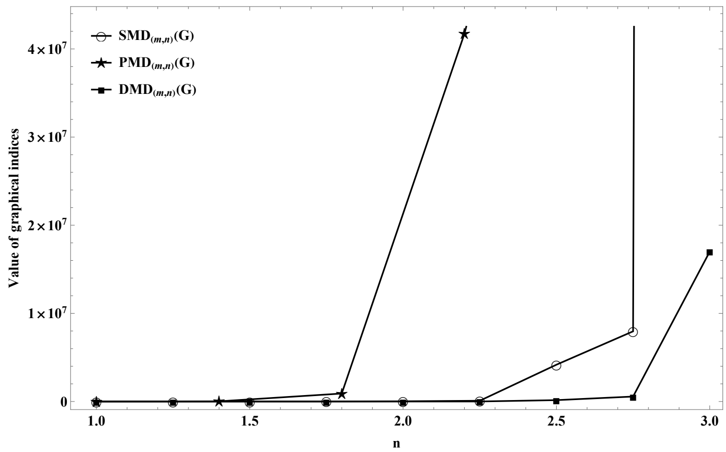

Figure 3: The comparative analysis of G_2.

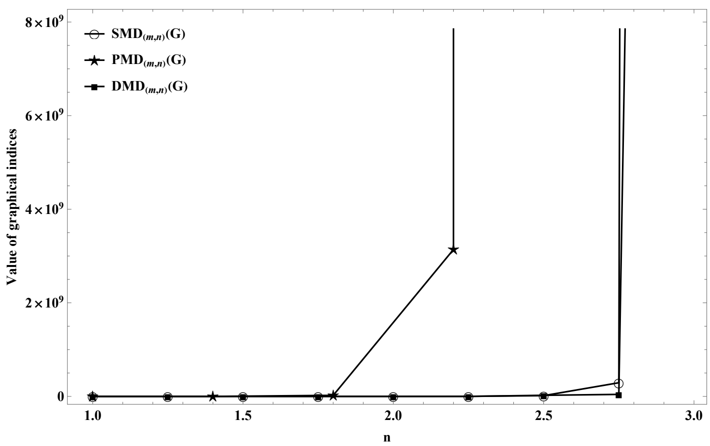

Figure 4: The comparative analysis of G_3.

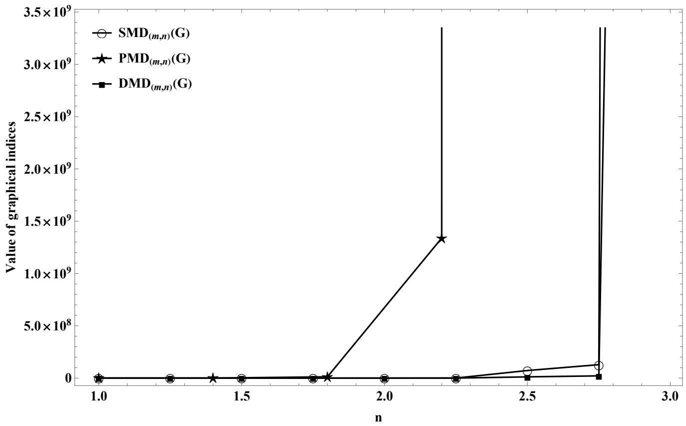

Figure 5: The comparative analysis of G_4.

6. Comparative Analysis:

Given the particular values of the (m, n)-minimal dominating graphical indices of the molecular graph of some basic Benzenoid structures as shown in Table 4 for $1 \leqslant \{m, n\} \leqslant 3$, we have the comparative analysis among the $SMD_{(m,n)}(G_i)$, $PMD_{(m,n)}(G_i)$ and $DMD_{(m,n)}(G_i)$ of molecular graph of some basic Benzenoid structures G_i for $1 \leqslant i \leqslant 4$ as shown in Figure 2 to Figure 5 as follows:

(i) In G_1, the value of $SMD_{(m,n)}(G_1)$ with $m = n \geqslant 1$ is $12n$; $n \geqslant 1$, the value of $PMD_{(m,n)}(G_1)$ with $m = n \geqslant 1$ is stagnate at the value 6. But $DMD_{(m,n)}(G_1)$ doesnot exist.

(ii) In G_i, $DMD_{(m,n)}(G_i) \leqslant SMD_{(m,n)}(G_i) \leqslant PMD_{(m,n)}(G_i)$ for $i = 2, 3, 4$ and $\{m, n\} \geqslant 1$.

7. Conclusion and Open Problems

In this paper, the classical concepts of domination-related parameters and graphical indices are combined and initiated to study the generalized minimal dominating graphical indices, which lie on the claim that their particular cases, for pertinently chosen values of two real numbers m and n. Here, we have the following Open problems.

(i) Obtain the some bounds and characterization among the (m, n)-minimal dominating graphical indices namely $SMD_{m,n}(G)$,
$PMD_{m,n}(G)$ and $DMD_{m,n}(G)$.

(ii) Obtain some bounds and characterization of (m, n)-minimal dominating graphical indices in terms of other graph theoretical parameters such as covering and independence number of a graph.

(iii) Find some results on (m, n)-minimal dominating graphical indices of certain families of derived graphs/ transformation graphs /product graphs.

(iv) Find the values of the (m, n)-minimal dominating graphical indices of certain classes of chemical graphs and compare them with degree/distance/spectral-based graphical indices. Also, explore some results towards the QSPR / QSAR / QSTR Model.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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